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Classical and quantum temperature fluctuations via holography

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Abstract

We study local temperature fluctuations in a 2+1 dimensional CFT on the sphere, dual to a black hole in asymptotically AdS space-time. The fluctuation spectrum is governed by the lowest-lying hydrodynamic sound modes of the system whose frequency and damping rate determine whether temperature fluctuations are thermal or quantum. We calculate numerically the corresponding quasinormal frequencies and match the result with the hydrodynamics of the dual CFT at large temperature. As a by-product of our analysis we determine the appropriate boundary conditions for calculating low-lying quasinormal modes for a four-dimensional Reissner-Nordström black hole in global AdS.

May 19, 2014

1 Introduction

The field of black hole thermodynamics was cast into life by the definition of black hole entropy and temperature in the seminal papers by Bekenstein [1] and Hawking [2]. Early on it was viewed as an interesting analogy between dynamical equations of two rather different fields of physics, general relativity and statistical mechanics, but with the advent of anti de Sitter - conformal field theory (AdS/CFT) duality [3] the analogy has been promoted to a precise correspondence for a class of black holes.

A Schwarzschild black hole in asymptotically flat spacetime is unstable due to the Hawking effect. It evaporates if it is surrounded by empty spacetime and as it has negative specific heat it cannot be in stable equilibrium with a thermal gas either. In fact, due to the Jeans instability, the thermodynamic limit is not well defined in Einstein gravity in asymptotically flat spacetime and there exist no equilibrium configurations at finite temperature and density.

The situation is different in asymptotically AdS spacetime, where a large black hole above the Hawking-Page phase transition [4] is a stable configuration. Under AdS/CFT duality, such a black hole corresponds to a thermal state in the conformal field theory at the boundary of spacetime, with the CFT temperature equal to the Hawking temperature of the black hole [5]. The appropriate interpretation of black hole thermodynamics in asymptotically AdS spacetime is in terms of the dual field theory rather than spacetime physics in the bulk. The AdS black hole geometry itself is a solution of the classical field equations of general relativity without matter and observers in free fall outside a large AdS black hole do not detect any propagating Hawking radiation [6]. Nonetheless the temperature of the dual CFT is a well-defined observable.

Any macroscopic physical observable is subject to statistical fluctuations. When one measures a particular quantity one gets results which are distributed around the mean value with some finite standard deviation. For the macroscopic objects the fluctuations are usually governed by random thermal noise, but when the system is small enough, the quantum uncertainty priciple starts playing a significant role and renders the overall behavior of the system quite different. Thus one distinguishes between the regimes of thermal and quantum fluctuations of the observable. This treatment can be readily applied to the fluctuations of the temperature of the system, which we are considering here. Due to energy conservation, the total value of the temperature in the dual field theory does not fluctuate. However, local fluctuations do exist and have a well-defined physical meaning. So in the system under consideration one should be able to observe thermal or quantum regimes of the fluctuations of local temperature.

In this paper we use AdS/CFT duality to investigate local temperature fluctuations in a field theoretic system at strong coupling via the dynamics of asymptotically AdS black holes. We focus on the AdS-Reissner-Nordström solution dual to a CFT at finite temperature and chemical potential. The rationale for switching on a chemical potential is that it allows us to consider low temperature without compromising thermodynamic stability, which as we shall see leads to a crossover from a thermal to a quantum regime in temperature fluctuations.

The paper is organized as follows. In Section 2 we recall the notion of temperature fluctuations and analyze them in the case of underdamped modes. In Section 3 we consider the hydrodynamic approximation to the sound modes of CFT on the sphere, dual to the black hole under consideration, and in Section 4 we carry out a direct gravitational analysis of the system and compute numerically the lowest-lying quasinormal modes of the black hole. We conclude in Section 5. Appendix A is devoted to the detailed analysis of the transition between classical and quantum regimes of the fluctuations. In Appendix B we consider details of the calculation of QNM of Reissner-Nordström black hole in global AdS_4.

2 Temperature fluctuations

Temperature fluctuations in classical systems have been widely studied. In classical statistical mechanics the variance of a statistical variable is given by the width of its probability distribution, which for temperature gives [7][1]

$$\langle \delta T^2 \rangle = \frac{T^2}{C_v}, \tag{1}$$

where T is the temperature and C_v is the specific heat. From this expression, it is clear that in order for the temperature to be a well-defined quantity, the specific heat must be large.

Temperature fluctuations close to equilibrium can be described by linear response theory [8, 9]. To this end, we consider a system in which the temperature of a black body is in equilibrium with the surrounding radiation. In response to an external perturbation the system will relax to equilibrium on a characteristic time-scale τ, with its temperature changing with time according to

$$\frac{dT}{dt} = -\frac{T - T_e}{\tau}, \tag{2}$$

with T_e being the equilibrium temperature. The change in entropy ΔS caused by a perturbation is related to the change in temperature as

$$\Delta T_e = \frac{\partial T}{\partial S}\Delta S = \frac{T}{C_v}\Delta S. \tag{3}$$

The Fourier spectrum of temperature fluctuations is thus related to that of the entropy by a response function (generalized susceptibility):

$$\delta T(\omega) = \alpha(\omega)\Delta S(\omega), \tag{4}$$

given by (3) and (2):

$$\alpha(\omega) = \frac{T}{C_v}\frac{1}{1 - i\omega\tau}\,. \tag{5}$$

The fluctuation-dissipation theorem [7] relates the mean square temperature fluctuation to the imaginary part of the susceptibility:

$$\langle \delta T^2 \rangle = \hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ \mathrm{Im}\,\alpha(\omega) \coth\frac{\hbar\omega}{2T}\,. \tag{6}$$

The divergence of the integral at high frequencies is fictitious, and is an artifact of the single-pole form for the response function, which is just an approximation valid at low frequencies. The integral

[1] We set $k_B = 1$ but will keep the explicit dependence on $\hbar$ in this section and in App. A for distinguishing classical and quantum contributions.

cannot be carried out exactly, but it can easily be approximated in various limits [9]. In particular the classical result (1) is reproduced at $T\tau \gg \hbar$, by replacing the coth by the inverse of its argument. In the opposite regime of $T\tau \ll \hbar$,

$$\langle \delta T^2 \rangle \simeq \frac{\hbar T}{\pi C_v \tau} \ln \omega_c \tau, \tag{7}$$

where ω_c is a cutoff frequency, see App. A for details.

Thus far we have been working in the approximation in which the dominant relaxation is given by an overdamped mode. In studying temperature fluctuations of the AdS black hole we will encounter a different situation, when relaxation to equilibrium is driven by slowly decaying oscillations. The above discussion then needs to be modified. To this end, we consider the damped harmonic oscillator with an internal frequency Ω and relaxation rate Γ. The response function, or the retarded Green function, is given by

$$\alpha(\omega) = G^R(\omega) = -\frac{\Omega^2}{k} \frac{1}{\omega^2 - \Omega^2 + i2\omega\Gamma}. \tag{8}$$

Here k is the "spring constant" which in our case is given by $k = \frac{C_v}{T}$ since the change in free energy is $\Delta S \delta T = \frac{C_v}{T} \delta T^2$. So the response function is given by

$$\alpha(\omega) = -\frac{T\Omega^2}{C_v} \frac{\omega^2 - \Omega^2 - i2\omega\Gamma}{(\omega^2 - \Omega^2)^2 + 4\omega^2\Gamma^2}. \tag{9}$$

The two oscillation poles in the complex frequency plane are located at

$$\omega = -i\Gamma \pm \sqrt{\Omega^2 - \Gamma^2} \simeq -i\Gamma \pm \Omega.$$

The approximate equality on the right holds for $\Omega \gg \Gamma$, the case that we will encounter shortly in the holographic setup.

Depending on the ratios between the three parameters T, Ω and Γ, various regimes of temperature fluctuations are possible. In particular, if temperature is much bigger than both the frequency and decay rate, the classical result (1) is recovered. The leading quantum corrections can be readily computed, here for simplicity displayed in the oscillatory regime $T/\hbar \gg \Omega \gg \Gamma$ (see App. A):

$$\langle \delta T^2 \rangle \simeq \frac{T^2}{C_v} + \frac{\hbar^2\Omega^2}{12C_v} - \frac{\hbar^3\zeta(3)\Omega^2\Gamma}{2\pi^3 C_v T} + \mathcal{O}\left(\frac{\hbar^4\Omega^2\Gamma^2}{T^2}\right). \tag{10}$$

The first two terms come from the poles in the response function and the third term is due to a summation of Matsubara modes.

The "quantum" regime comes about when $T/\hbar \ll \max(\Omega, \Gamma)/2\pi$, which when $\Omega \gg \Gamma$ yields

$$\langle \delta T^2 \rangle \simeq \frac{\hbar T \Omega}{2C_v}. \tag{11}$$

A careful analysis is carried out in App. A where we define the classicality parameter (for $\Omega > \Gamma$):

$$\mathfrak{q} \equiv \frac{2\pi T}{\hbar\Omega}. \tag{12}$$

If $\mathfrak{q} > 1$ the temperature fluctuations are in the classical regime, while for $\mathfrak{q} < 1$ the temperature fluctuations are quantum.

The overdamped regime is recovered when $\Gamma \gg \Omega$. Then a new scale emerges, namely $\tau = \Gamma/\Omega^2$, which plays the rôle of the relaxation time. In this regime the temperature fluctuations obey

$$\langle \delta T^2 \rangle \simeq \frac{\hbar T}{\pi C_v \tau} \log \Gamma\tau \,, \tag{13}$$

which coincides with (7) provided that the cutoff frequency ω_c is identified with Γ.

3 Temperature fluctuations of charged black hole in hydrodynamic approximation

The system we are interested in is a black hole in the space of constant negative curvature (AdS_{d+2}), which is dual to a strongly-coupled CFT in $d+1$ dimensions, heated to a temperature that coincides with the Hawking temperature of the black hole. For the most part, we consider the four-dimensional black-hole ($d = 2$), but many formulas in this section are valid in any number of dimensions. Since the neutral AdS black hole becomes thermodynamically unstable at low temperatures, we shall consider a larger class of solutions, namely AdS-Reissner-Nordström black holes which carry non-zero charge and are dual to a CFT at non-zero chemical potential. This will allow us to probe the low-temperature regime when quantum effects are expected to be important.

The metric of the AdS_4 black hole under consideration is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{14}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}. \tag{15}$$

The temperature, entropy, chemical potential[2] and extremal charge of the black hole are given by

$$T = \frac{1 + 3\frac{r_+^2}{R^2} - \frac{Q^2}{r_+^2}}{4\pi r_+} = \frac{1 + 3\frac{r_+^2}{R^2}}{4\pi r_+}\left(1 - \frac{Q^2}{Q_{\mathrm{ext}}^2}\right), \qquad S = \pi r_+^2, \qquad \mu = \frac{Q}{r_+}, \qquad Q_{\mathrm{ext}} = r_+\sqrt{1 + 3\frac{r_+^2}{R^2}}, \tag{16}$$

where r_+ is the horizon radius, defined as the largest root of $f(r_+) = 0$. We absorb the Planck mass M_{pl} into the definition of the parameters M and Q, which now have the dimension of length. The specific heat of the black hole is given by [10]

$$C_v = 2\pi r_+^2 \left(\frac{3r_+^2 - R^2}{3r_+^2 + R^2} + \frac{Q^2}{Q_{\mathrm{ext}}^2}\right)^{-1}\left(1 - \frac{Q^2}{Q_{\mathrm{ext}}^2}\right), \tag{17}$$

which vanishes, along with the temperature, for the extremal black hole.

The dual CFT is defined on $S^2 \times \mathbb{R}_t$, the sphere has radius R, because far away from the horizon the metric (14) asymptotes to

$$ds^2 \simeq \frac{r^2}{R^2}\, ds^2_{\mathrm{boundary}} + \frac{R^2}{r^2}\, dr^2, \qquad ds^2_{\mathrm{boundary}} = -dt^2 + R^2(d\theta^2 + \sin^2\theta \, d\varphi^2). \tag{18}$$

[2] This is the electrostatic potential of the black hole, identified with the chemical potential of the dual field theory by the AdS/CFT correspondence.

We will be interested in the statistics of temperature fluctuations of the dual field theory on the boundary of spacetime. Let $\delta T(\theta,\varphi)$ be the difference between the local temperature at the point (θ,φ) on S^2 and the Hawking temperature. It is convenient to expand the temperature difference in spherical harmonics:

$$\delta T(\theta,\varphi)=\sum_{lm}\delta T_{lm}Y_{lm}(\theta,\varphi). \tag{19}$$

The spherical functions are assumed to be canonically normalized:

$$\int_{S^d}d^dx\,\sqrt{g}\,Y^*_{lm}(x)Y_{l'm'}(x)=V\delta_{ll'}\delta_{mm'}, \tag{20}$$

where V is the surface area of the sphere. The two-point correlation function of temperature fluctuations, by rotational symmetry, should be independent of the magnetic quantum numbers. We thus define the power spectrum of temperature fluctuations in the l-th harmonics as

$$\langle\delta T^*_{lm}\delta T_{l'm'}\rangle\equiv\left\langle\delta T_l^2\right\rangle\delta_{ll'}\delta_{mm'}. \tag{21}$$

The inverse Hawking temperature is identified with the Euclidean-time periodicity of the black hole solution, which means that the black hole temperature fluctuations are related to fluctuations of the g_{00} metric component. This relationship can be made precise with the help of the AdS/CFT correspondence. According to AdS/CFT, metric fluctuations are dual to the energy-momentum tensor in the field theory on the boundary of AdS. The local temperature variation, in any CFT, is related to the local variation in the energy density:

$$\delta T=\frac{1}{c_v}\,\delta T_{00}, \tag{22}$$

where $c_v=\partial\epsilon/\partial T$ is the volumetric heat capacity, and $T_{\mu\nu}$ is the energy-momentum tensor of the CFT. The fluctuation-dissipation theorem then expresses the power spectrum of temperature fluctuations through the retarded density-density correlator:

$$\begin{aligned}\left\langle\delta T_l^2\right\rangle &= \frac{1}{c_v^2V}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\operatorname{Im}G_l(\omega)\coth\frac{\omega}{2T} & (23)\\ G_l(\omega) &= i\int_0^\infty dt\,\mathrm{e}^{\,i\omega t}\int d^dx\,\sqrt{g}\,Y_{l0}(x)\left\langle[T_{00}(t,x),T_{00}(0,0)]\right\rangle. & (24)\end{aligned}$$

The factor of $1/V$ in the first equation arises because of the normalization (20) of the spherical functions.

The two-point correlator of the energy density can be calculated holographically by studying the response of the gravitational background to scalar metric perturbations. The retarded two-point function is then expressed in terms of the quasinormal modes (QNMs) of the black hole [11, 12]. For a black hole with a flat horizon, the lowest QNMs exhibit hydrodynamic behavior consistent with shear and sound modes of the thermalized plasma state of the dual CFT [11, 12]. The hydrodynamic approximation should still be accurate for a sufficiently large black hole with a spherical horizon. Indeed one can calculate the lowest QNMs of a large AdS black hole from the hydrodynamics on the sphere, without any recourse to Einstein's equations [13, 14]. In this section we compute the response function in the same hydrodynamic approximation, which should be valid in the high-temperature regime, $TR\gg 1$.

The hydrodynamic equations of motion follow from the conservation of the energy-momentum tensor:

$$\nabla_\mu T^{\mu\nu} = 0, \tag{25}$$

where

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p\delta^{\mu\nu} - \eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\nabla_\mu u^\mu\right) - \zeta\Delta^{\mu\nu}\nabla_\lambda u^\lambda. \tag{26}$$

Here u_μ is the local 4-velocity of the liquid (satisfying $u^\mu u_\mu = -1$), ϵ is its energy density, p is the pressure, η and ζ are the shear and bulk viscosities, and the covariant derivative ∇_μ is taken with respect to the boundary metric (18). For a conformal theory, $T^\mu_\mu = 0$ leads to $\zeta = 0$ and $\epsilon = dp$.

The standard way to compute the two-point function of the energy-momentum tensor is to study a response to metric perturbations. Then,

$$\left\langle T^{\mu\nu}(x)\right\rangle_{g+h} = \left\langle T^{\mu\nu}(x)\right\rangle_g + \frac{i}{2}\int d^{d+1}y\,\sqrt{|g|}\left\langle T^{\mu\nu}(x)T^{\lambda\rho}(y)\right\rangle_g h_{\lambda\rho}(y) + O\left(h^2\right). \tag{27}$$

To find the density-density correlation function we thus need to linearize the hydrodynamic equations in the presence of a small lapse function h_{00} on top of the metric of $S^d \times \mathbb{R}_t$. The linearized Navier-Stokes equations on $S^d \times \mathbb{R}_t$ can be found in [13, 14]. Keeping track of the non-zero lapse function, we can recover the source term. This results in a coupled system of two linear equations:

$$\begin{pmatrix} \partial_t & (\epsilon+p) \\ c_s^2\nabla^2 & (\epsilon+p)\,\partial_t - \frac{2}{d}\,\eta\mathcal{R} - \left(\zeta + 2\,\frac{d-1}{d}\,\eta\right)\nabla^2 \end{pmatrix}\begin{pmatrix} \delta\epsilon \\ \nabla_i\delta u^i \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\,(\epsilon+p)\,\nabla^2 h_{00} \end{pmatrix}, \tag{28}$$

where $\mathcal{R} = d(d-1)/R^2$ is the Ricci curvature and $-\nabla^2$ is the invariant Laplacian on the sphere, and $c_s^2 = \partial p/\partial\epsilon$ is the speed of sound. Expanding in spherical harmonics, solving for $\delta\epsilon$ and comparing with (27), we get for the Green's function defined in (24):

$$G_l(\omega) = -\frac{\epsilon+p}{c_s^2}\,\frac{\Omega_l^2}{\omega^2 - \Omega_l^2 + 2i\omega\Gamma_l} \tag{29}$$

with

$$\Omega_l = \frac{c_s}{R}\sqrt{l\,(l+d-1)} \tag{30}$$

$$\Gamma_l = \frac{1}{(\epsilon+p)\,R^2}\left[\frac{(d-1)\,(l+d)\,(l-1)}{d}\,\eta + \frac{l\,(l+d-1)}{2}\,\zeta\right]. \tag{31}$$

The correct normalization of the response function, as in (9), follows from (23), (29) by virtue of a thermodynamic identity

$$c_v c_s^2 = \frac{\partial\epsilon}{\partial T}\,\frac{\partial p}{\partial\epsilon} = \frac{\partial p}{\partial T} = s = \frac{\epsilon+p}{T}.$$

This guarantees matching to classical thermodynamics (1) in the high-temperature limit.

Taking into account that for a conformal fluid, $c_s^2 = 1/d$ and $\zeta = 0$, and using the universal holographic result [15, 11] for the viscosity-to-entropy ratio $\eta/s = 1/4\pi$ gives for the hydrodynamic QNMs [13, 14]:

$$\omega_{\rm hyd} = \pm\frac{1}{R}\sqrt{\frac{l\,(l+d-1)}{d}} - \frac{i(d-1)(l+d)(l-1)}{4\pi d T R^2}. \tag{32}$$

We have taken into account here that $\Omega_l/\Gamma_l \sim RT/l \gg 1$ (unless l is very big, but the hydrodynamic approximation is not applicable to such high-frequency modes anyway), and so the sound modes attenuate very slowly. It is also true that $T/\Omega_l \sim TR/l \gg 1$, which implies that temperature fluctuations are purely classical as long as hydrodynamics is an accurate approximation.

Our discussion so far applied to neutral black holes. The main complication that arises for charged black holes is that eq. (22) does not hold true any more. The energy density of a charged fluid is a function of two variables, the temperature and chemical potential. Temperature fluctuations then mix with those in the chemical potential. As a result, δT is a linear combination of δT_{00} and the charge density fluctuation δJ_0. The response functions for temperature relaxation then includes, in addition to $\langle T_{00}T_{00}\rangle$, also a contribution from $\langle T_{00}J_0\rangle$ and $\langle J_0 J_0\rangle$. We will not discuss these complications further, so our results for charged black holes literally apply to fluctuations in energy rather than temperature.

However, equations (29)–(31) apply to a charged black hole without modifications. This is because the pressure of a conformal fluid is fixed by conformal symmetry, and thus charge and energy density fluctuations do not mix. Using the thermodynamic relation $\epsilon + p = Ts + \mu\rho$, we get:

$$\omega_{\rm hyd} = \pm\frac{1}{R}\sqrt{\frac{l(l+d-1)}{d}} - \frac{i(d-1)(l+d)l(l-1)}{4\pi d R^2}\frac{1}{T+\mu\frac{Q}{S}},$$

where S and Q are the total entropy and charge of the system. Substituting the values for temperature, charge, potential and entropy of the dual Reissner-Nordström black hole (16), we obtain the following final result for the sound mode frequency in the hydrodynamic regime

$$\omega_{\rm hyd} = \pm\frac{1}{R}\sqrt{\frac{l(l+d-1)}{d}} - \frac{i(d-1)(l+d)(l-1)}{d}\frac{r_+}{R^2+3r_+^2+3\mu^2R^2}. \tag{33}$$

As an illustration, the hydrodynamic quasinormal frequencies for the harmonics $l=2$ through $l=7$ at various values of r_+ and μ are shown for $d=2$ (i.e. AdS_4) in Fig. 1.

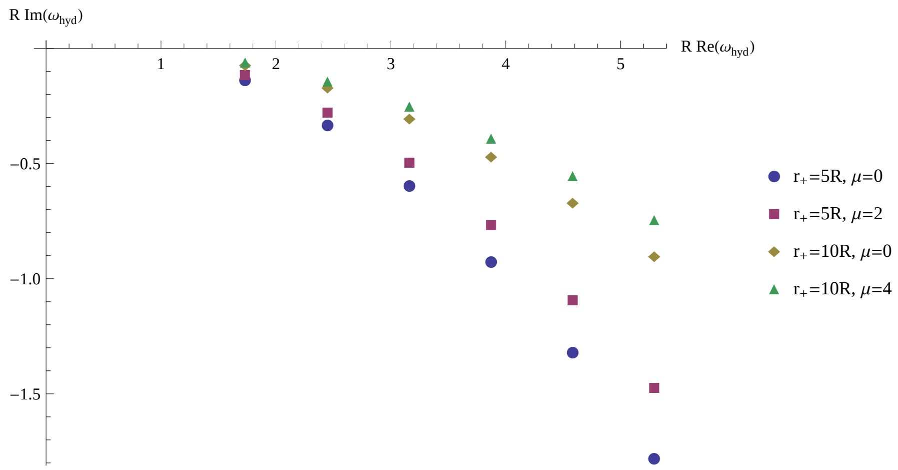

Figure 1: The harmonics with $l=2$ through $l=7$ of hydrodynamic quasinormal modes (33) at various values of r_+ and μ for the AdS-Reissner-Nordström black hole.

It is important to note here, that there are no quasinormal modes with $l=0$ and $l=1$ in the spectrum. The former would correspond to the homogeneous change of the energy density in the whole volume of the system and thus this fluctuation would violate the energy conservation law. From the point of view of linear response theory this means that once the system is perturbed by a force, which homogeneously changes the energy density, it acquires a new equilibrium state and does not relax to the initial one. The mode with $l = 1$ corresponds to a simple rotation in S^2. In the case of flat space it would correspond to a translation along a given direction. Thus in homogeneous space it is the Goldstone mode, associated with translation symmetry. Indeed the shift of the center of mass of the system will not change its state and hence not produce any counter-force, so there is no relaxation associated with this mode. The argument above indicates that we are dealing with *local* temperature fluctuations. The fluctuations of the total temperature may be described only by the $l = 0$ mode, because all the others average to zero upon integration over the volume of the system. But at the same time for the closed system with conserved energy, such as the large AdS-RN black hole or the CFT on a sphere, the fluctuations of the total temperature are forbidden by the energy conservation law and thus the $l = 0$ mode is absent. Nonetheless, *local* fluctuations of temperature are allowed and their study is a well-defined problem.

Finally, we note that for high temperatures, corresponding to large black hole radii, r_+, the imaginary part of the QNM frequency (33) is small and thus temperature fluctuations in this regime are always classical. In order to approach the quantum regime, we need to consider small r_+, which in turn is outside of the applicability of hydrodynamics. For this we will turn to the direct gravitational study of spherical black holes in the next section, for which the analysis necessitates numerical calculations.

4 Quasinormal modes of spherical black hole

The quasinormal modes for a spherical AdS-Reissner-Nordström black hole were thoroughly studied in [16, 17, 18, 19, 20] (see [21] for a review). The entire spectrum of quasinormal modes of the Reissner-Nordström black hole in AdS_4 was calculated in [20], following the approach developed in [18, 19, 22]. Although in the axial gravitational channel an "exceptional" frequency was found, which can be related to the hydrodynamic shear mode [11], no long-lived modes were observed in the polar gravitational channel, which would correspond to the sound mode [12] discussed above. This discrepancy with the hydrodynamic results was pointed out in [13, 14]. It turned out, that one should pay special attention to the boundary conditions of the polar gravitational mode, because simple Dirichlet boundary conditions on the master field lead to metric fluctuations, which perturb the asymptotic behavior of AdS_4 spacetime. Instead, special Robin boundary conditions were found, which do not perturb the asymptotic metric and lead to quasinormal modes consistent with the hydrodynamic picture. The corresponding numerical calculations were done for the 5- and 4-dimensional AdS-Schwarzschild metric in [13, 14] and analytic results for long-lived modes of neutral black holes in AdS-space of any dimension were obtained in [23]. The analogous treatment of Kerr-AdS black holes was made in [24, 25]. The quasinormal frequencies of the sound mode in an AdS-Reissner-Nordström black hole were obtained in [26] for AdS_{5-7} but have not been considered for AdS_4 to the best of our knowledge.[3] We address this problem in Appendix B and use the results here.

We study the quasinormal frequencies ω_0 of the gravitational polar mode (related to the dual hydrodynamic sound mode) for different choices of charge and horizon radii of the black hole. The resulting frequencies at $r_+ = 5R$ and $r_+ = 10R$ for angular momentum $l = 2$ are shown in Fig. 2. The curves demonstrate similar behavior to that observed for the "exceptional" mode in the axial channel

[3] We note also, that in [26] Dirichlet boundary conditions were used for the master function at the AdS boundary and quasinormal modes calculated in this way may not coincide with the hydrodynamic ones.

in [20]. For a neutral black hole our results coincide with those presented in [14] for $r_+ \lesssim 20R$, which is the range over which our numerical calculation yields reliable precision.

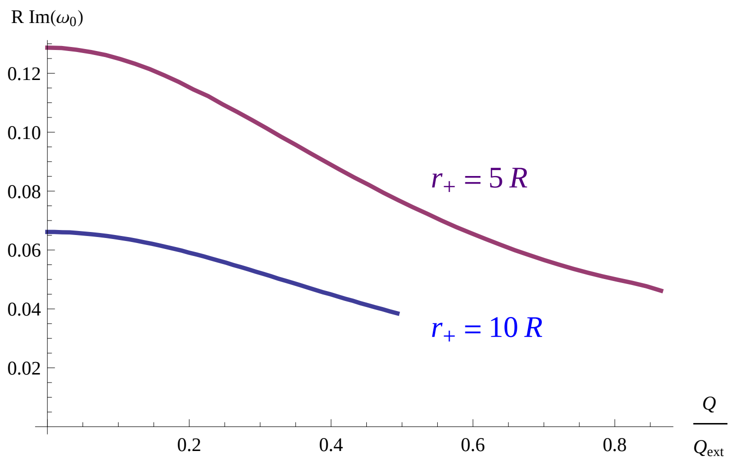

Figure 2: Imaginary part of the quasinormal frequency of the gravitational mode with $l = 2$ for $r_+ = 5R$ and $r_+ = 10R$ as a function of the black hole charge. The frequency is given in units of the AdS curvature-radius R.

At finite electrostatic potential it is especially interesting to compare the results of our computation to the hydrodynamic approximation (33). Fig. 3 shows the relation between the numerical result and the hydrodynamic one for different black hole temperatures. One can see that the curves approach unity quite fast and already at $T \approx 2R^{-1}$ the discrepancy is less than one percent, which is comparable to our numerical precision. This is a valuable check of the applicability of our procedure for calculating quasinormal modes.

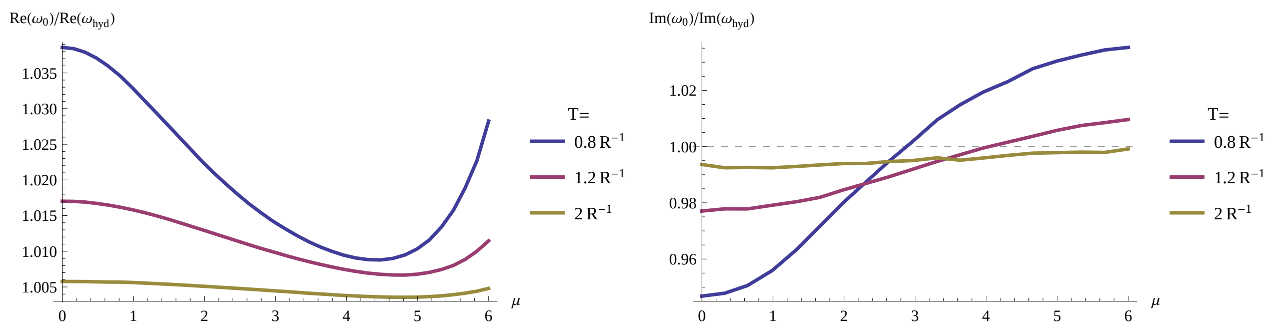

Figure 3: The relative values of the real and imaginary part of the quasinormal frequency at $l = 2$ with respect to that given in the hydrodynamic approximation (33).

Using the numerical values of the quasinormal frequencies for various T and μ, we can calculate the value of the parameter $\mathfrak{q}$, which measures the classicality of the temperature fluctuations (12). It is instructive to plot the curve $\mathfrak{q} = 1$ in the phase diagram of the charged black hole in AdS space in the grand canonical ensemble, i.e. for fixed temperature T and asymptotic electrostatic potential μ.

The phase diagram in Fig. 4 contains also the phase transition between the Reissner-Nordström black hole and the thermal gas of particles in the background of the extremal black hole with given potential μ, denoted by AdS* [27]. At zero charge this phase transition reduces to the Hawking-Page transition for the AdS-Schwarzschild black hole [4]. We find that the lowest value of the parameter $\mathfrak{q}$, which is achieved before this phase transition occurs, is $\mathfrak{q} \approx 0.92$ so the deep quantum regime is never reached for the neutral black hole. Instead one should speak about the region where quantum effects are already important. It is worth noting that the line $\mathfrak{q} = 1$ presumably asymptotes to the linear relation between temperature and potential, so concerning the classical/quantum transition in the behavior of the temperature fluctuations, T/μ is the relevant parameter, rather than T itself. This argument is consistent with the conformal nature of the system under consideration, where only the relation between temperature and chemical potential has physical meaning. The other interesting line in the phase diagram is the one describing the region, where the temperature (energy) fluctuations computed numerically via (6) using (9) for the gravitational quasinormal modes are of the order of the temperature itself. Linear response theory is no longer reliable below this line and it becomes meaningless to speak of small fluctuations. Extremal black holes are thus not included in our treatment.

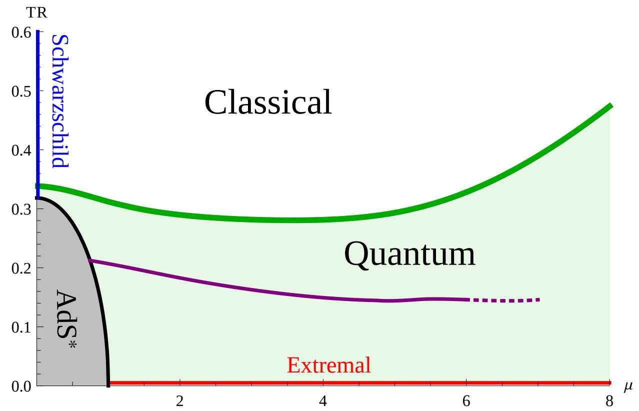

Figure 4: Phase structure of the charged black hole in AdS_4 in the grand canonical ensemble. The black line marks the "Hawking-Page type" phase transition [27]. The green line is $\mathfrak{q} = 1$, and thus in the green region quantum effects become important. The purple line marks the region, where the numerically computed fluctuation $\langle \delta T^2 \rangle^{1/2}$ is comparable to the temperature itself (at $\mu > 6$ our precision does not allow us to continue the line reliably).

5 Discussion

We have studied temperature (more precisely, energy) fluctuations in the holographic dual of the AdS-Reissner-Nordström black hole. Interestingly, temperature fluctuations are described by quasinormal modes of a black hole in the sound channel, for which the oscillation frequencies are much larger than the attenuation rate. This leads to a rather specific behavior of temperature fluctuations. For instance, the transition from thermal to quantum fluctuations is controlled by the oscillation frequency (related to the speed of sound) rather than damping rate, as it would have been for the shear modes.

As expected, temperature fluctuations behave classically for a large black hole and become more and more quantum as the black hole temperature is lowered. Temperature fluctuations of a neutral AdS black hole never become really quantum, because the black hole becomes thermodynamically unstable and undergoes the Hawking-Page phase transition before the fluctuations enter the quantum regime. Charged black holes can be sufficiently small for fluctuations to become quantum. For nearly-extremal black holes fluctuations become so strong that one presumably cannot trust the linear response theory any more. The strongly non-equilibrium behavior that sets in can perhaps be analyzed along the lines of [28, 29].

We reiterate that our analysis for charged black holes applies to the fluctuations of energy rather than temperature. The temperature fluctuations mix with those of charge density, which may give rise to an overdamped pole in the response function. We leave this interesting issue for future investigation.

Acknowledgments

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A Temperature fluctuations

A.1 The overdamped mode

Let us consider the integral

$$\langle \delta T^2 \rangle = \frac{\hbar T}{2\pi C_v} \int_{-\infty}^{\infty} d\omega \, \frac{\omega\tau}{(\omega\tau)^2+1} \coth\left(\frac{\hbar\omega}{2T}\right) = \frac{\hbar T}{2\pi C_v \tau} \int_{-\infty}^{\infty} dx \, \frac{x}{x^2+1} \coth\left(\frac{x}{2\mathfrak{r}}\right), \tag{34}$$

where $x = \omega\tau$ and we have defined

$$\mathfrak{r} \equiv \frac{T\tau}{\hbar}. \tag{35}$$

We can now conveniently define

$$I \equiv \frac{2\pi C_v \tau}{\hbar T} \langle \delta T^2 \rangle = \int_{-\infty}^{\infty} dx \, f, \qquad f \equiv \frac{x}{x^2+1} \coth\left(\frac{x}{2\mathfrak{r}}\right), \tag{36}$$

As I is UV-divergent, we introduce a cutoff $x_c \equiv \omega_c \tau$: $I_{\text{reg}} = \int_{-x_c}^{x_c} dx \, f$. Considering now the contour integral

$$I_\gamma = \oint_\gamma dz \, \frac{z}{z^2+1} \coth\left(\frac{z}{2\mathfrak{r}}\right), \tag{37}$$

where γ is given in fig. 5. The contribution from the arc will vanish for $x_c \to \infty$, i.e.

$$\lim_{x_c\to\infty} I_\gamma = \lim_{x_c\to\infty} \text{p.v. } I_{\text{reg}}, \tag{38}$$

as long as the contour does not hit a pole on the imaginary axis.

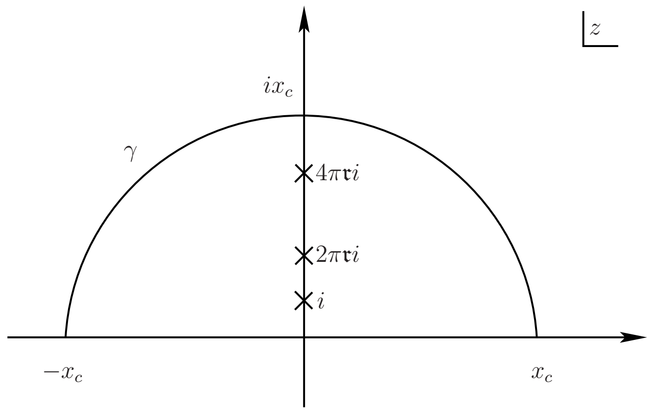

Figure 5: Contour γ in the complex x plane.

Using residue theory, we have

$$I_\gamma = 2\pi i \left[\mathrm{Res}(f, i) + \sum_{n=1}^{\Lambda} \mathrm{Res}(f, 2n\pi\mathfrak{r}i) \right], \qquad \Lambda \equiv \mathrm{floor}\left(\frac{x_c}{2\pi\mathfrak{r}}\right). \tag{39}$$

The residues can be calculated easily

$$\mathrm{Res}(f, i) = -\frac{i}{2}\cot\left(\frac{1}{2\mathfrak{r}}\right), \tag{40}$$

$$\mathrm{Res}(f, 2n\pi\mathfrak{r}i) = \frac{4n\pi\mathfrak{r}^2 i}{1 - 4n^2\pi^2\mathfrak{r}^2}, \tag{41}$$

and thus the contour integral reads

$$I_\gamma = \pi\cot\left(\frac{1}{2\mathfrak{r}}\right) - \sum_{n=1}^{\Lambda} \frac{8n\pi^2\mathfrak{r}^2}{1 - 4n^2\pi^2\mathfrak{r}^2}. \tag{42}$$

So far we did not assume anything about $\mathfrak{r}$ other than it being a real positive constant. In the limit $\mathfrak{r} \gg 2\pi$, we can expand the above expression to get

$$I_\gamma \simeq 2\pi\mathfrak{r} + 2\sum_{n=1}^{\Lambda} \frac{1}{n}, \tag{43}$$

where the last term is the harmonic sum, which is known to be divergent. Expressing the harmonic sum in terms of the cutoff, we have

$$I_\gamma \simeq 2\pi\mathfrak{r} - \frac{\pi}{6\mathfrak{r}} + 2\log\left(\frac{x_c}{2\pi\mathfrak{r}}\right) + 2\gamma_E + \mathcal{O}\left(\frac{\mathfrak{r}}{x_c}, \frac{1}{\mathfrak{r}^3}\right), \tag{44}$$

where γ_E is the Euler constant. Finally, we can write [9]

$$\langle \delta T^2 \rangle = \frac{T^2}{C_v}\left[1 - \frac{\hbar^2}{12T^2\tau^2} + \frac{\hbar}{\pi T\tau}\left(\log\left(\frac{\hbar\omega_c}{2\pi T}\right) + \gamma_E\right) + \mathcal{O}\left(\frac{1}{\omega_c\tau}, \frac{\hbar^4}{(T\tau)^4}\right)\right], \tag{45}$$

where ω_c is the cut-off frequency.

In the opposite limit, $\mathfrak{r} \ll 2\pi$, the temperature fluctuations are in the quantum regime and can be approximated as follows. We neglect the residues coming from the response function and approximate the sum in (42) with an integral and obtain

$$I_\gamma \simeq \log x_c^2, \tag{46}$$

which we can write as [9]

$$\langle \delta T^2 \rangle \simeq \frac{\hbar T}{\pi C_v \tau} \log \omega_c \tau. \tag{47}$$

A more careful treatment of the arc is necessary because a simple power counting naively tells us that it is logarithmically divergent. We want to show that

$$\lim_{x_c \to \infty} \int_{C_{x_c}^+} dz \, \frac{z}{z^2+1} \coth\left(\frac{z}{2\mathfrak{r}}\right) = 0, \tag{48}$$

where $C_{x_c}^+$ denotes the northern semicircle of radius x_c. The arc can be parametrized by $z = x_c e^{i\theta}$, where $\theta \in [0, \pi]$. Rewriting the coth, we have

$$\coth\left(\frac{z}{2\mathfrak{r}}\right) = \left(1 + \frac{2e^{-z/\mathfrak{r}}}{1-e^{-z/\mathfrak{r}}}\right) = \left(-1 - \frac{2e^{z/\mathfrak{r}}}{1-e^{z/\mathfrak{r}}}\right), \tag{49}$$

which we will use for the first and second quadrants of the z plane, respectively. We therefore have

$$\begin{aligned}\int_{C_{x_c}^+} dz \, \frac{z}{z^2+1} \coth\left(\frac{z}{2\mathfrak{r}}\right) &= \int_{C_{x_c}^{+1}} dz \, \frac{z}{z^2+1} \left(1 + \frac{2e^{-z/\mathfrak{r}}}{1-e^{-z/\mathfrak{r}}}\right) + \int_{C_{x_c}^{+2}} dz \, \frac{z}{z^2+1} \left(-1 - \frac{2e^{z/\mathfrak{r}}}{1-e^{z/\mathfrak{r}}}\right) \\ &= I_{+1}^{\text{pol}} + I_{+1}^{\text{exp}} + I_{+2}^{\text{pol}} + I_{+2}^{\text{exp}}.\end{aligned} \tag{50}$$

The integrals of the two polynomials in z can be carried out

$$I_{+1}^{\text{pol}} + I_{+2}^{\text{pol}} = i\int_0^{\frac{\pi}{2}} d\theta \, \frac{x_c^2 e^{i2\theta}}{x_c^2 e^{i2\theta}+1} - i\int_{\frac{\pi}{2}}^{\pi} d\theta \, \frac{x_c^2 e^{i2\theta}}{x_c^2 e^{i2\theta}+1} = 2\Re\left[i\int_0^{\frac{\pi}{2}} d\theta \, \frac{x_c^2 e^{i2\theta}}{x_c^2 e^{i2\theta}+1}\right] = -2\,\text{arccoth}(x_c^2), \tag{51}$$

which goes to zero in the limit of $x_c \to \infty$. Now let us consider the first integral with exponentials

$$I_{+1}^{\text{exp}} = \int_{C_{x_c}^{+1}} dz \, \frac{z e^{-z/\mathfrak{r}}}{(z^2+1)(1-e^{-z/\mathfrak{r}})}. \tag{52}$$

Since we are interested in the large x_c behavior of this integral, let us instead consider

$$\tilde{I}_{+1}^{\text{exp}} = \int_{C_{x_c}^{+1}} \frac{dz}{z} \, \frac{e^{-z/\mathfrak{r}}}{1-e^{-z/\mathfrak{r}}} = i\int_0^{\frac{\pi}{2}} d\theta \, \frac{\exp\left(-\frac{x_c}{\mathfrak{r}} e^{i\theta}\right)}{1-\exp\left(-\frac{x_c}{\mathfrak{r}} e^{i\theta}\right)}. \tag{53}$$

We will now need the lemma that

$$\left|\int_a^b dz \, f(z)\right| \leq \int_a^b dz \, M(z), \tag{54}$$

where $f(z)$ is a complex function and $M(z)$ is a real-valued function on the real interval $[a, b]$ such that $|f(z)| \leq M(z)$ everywhere on said interval (to prove this lemma, compose the integral into Riemann sums and use the triangle inequality). Hence we can write

$$|\tilde{I}_{+1}^{\exp}| \leq \int_0^{\frac{\pi}{2}} d\theta \, \frac{e^{-\frac{x_c}{\mathfrak{r}}\cos\theta}}{\sqrt{1 + e^{-\frac{2x_c}{\mathfrak{r}}\cos\theta} - 2e^{-\frac{x_c}{\mathfrak{r}}\cos\theta}\cos\left(\frac{x_c}{\mathfrak{r}}\sin\theta\right)}}. \tag{55}$$

To simplify the problem we will now assume that $x_c = (2m+1)\pi\mathfrak{r}$ with m being an integer. This value of the cut off is chosen such that the contour goes right in the middle between the Matsubara poles on the imaginary axis. We can now write

$$|\tilde{I}_{+1}^{\exp}| \leq A \int_0^{\frac{\pi}{2}} d\theta \, \exp\left(-\frac{x_c}{\mathfrak{r}}\cos\theta\right) = \frac{A\pi}{2}\left[I_0\big((2m+1)\pi\big) - L_0\big((2m+1)\pi\big)\right], \tag{56}$$

where $A \gtrsim 1$ is a constant of order one taking on the minimum value of the denominator of eq. (55). Expanding the above expression for large m, we have

$$|\tilde{I}_{+1}^{\exp}| \leq \frac{A}{(2m+1)\pi} + \mathcal{O}(m^{-3}), \tag{57}$$

which clearly goes to zero for $m \to \infty$. The same proof can be applied to $I_{+2}^{\exp}$ and hence is bounded by the same numerical value.

A.2 The underdamped mode

Let us now consider the integral

$$\langle \delta T^2 \rangle = \frac{\hbar T}{2\pi C_v}\int_{-\infty}^{\infty} d\omega \, \frac{2\omega\Gamma\Omega^2}{(\omega^2-\Omega^2)^2 + 4\omega^2\Gamma^2}\coth\left(\frac{\hbar\omega}{2T}\right) = \frac{\hbar T\Omega^2}{\pi C_v \Gamma}\int_{-\infty}^{\infty} dx \, \frac{x}{(x^2-\mathfrak{a}^2)^2 + 4x^2}\coth\left(\frac{x}{2\mathfrak{b}}\right), \tag{58}$$

where we have defined

$$x \equiv \frac{\omega}{\Gamma}, \qquad \mathfrak{a} \equiv \frac{\Omega}{\Gamma}, \qquad \mathfrak{b} \equiv \frac{T}{\hbar\Gamma}. \tag{59}$$

We will assume that $\mathfrak{a} > 1$. We can now conveniently define

$$I \equiv \frac{\pi C_v \Gamma}{\hbar T \Omega^2}\langle \delta T^2 \rangle = \int_{-\infty}^{\infty} dx \, f, \qquad f \equiv \frac{x}{(x^2-\mathfrak{a}^2)^2 + 4x^2}\coth\left(\frac{x}{2\mathfrak{b}}\right). \tag{60}$$

Notice that for this response function, the integral is no longer divergent. Consider now the contour integral

$$I_\gamma = \oint_\gamma dz \, \frac{z}{(z^2-\mathfrak{a}^2)^2 + 4z^2}\coth\left(\frac{z}{2\mathfrak{b}}\right), \tag{61}$$

where γ is given in fig. 6. The contribution from the arc will vanish for $x_c \to \infty$ and $x_c \neq 2\pi\mathfrak{b}n$, meaning that the cutoff does not make the contour cut through a pole, and thus

$$\lim_{x_c\to\infty} I_\gamma = \lim_{x_c\to\infty} \text{p.v. } I. \tag{62}$$

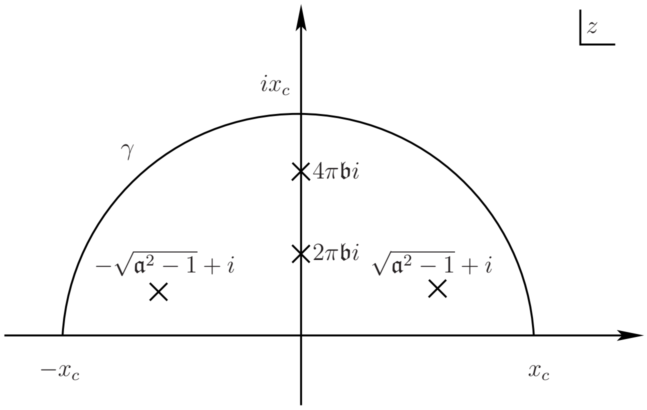

Figure 6: Contour γ in the complex plane.

It is easy to estimate that the contribution from the arc will be of the order of $1/x_c^2$.

We have two types of contributions, the first is due to the two classical poles at $\pm\sqrt{\mathfrak{a}^2-1}+i$ and the second comes from the n-th Matsubara mode at $2n\pi\mathfrak{b}i$,

$$I_\gamma = 2\pi i\left[\sum_{\pm}\mathrm{Res}(f, \pm\sqrt{\mathfrak{a}^2-1}+i) + \sum_{n=1}^{\Lambda}\mathrm{Res}(f, 2n\pi\mathfrak{b}i)\right], \qquad \Lambda \equiv \mathrm{floor}\left(\frac{x_c}{2\pi\mathfrak{b}}\right). \tag{63}$$

The residues can be calculated easily

$$\sum_{\pm}\mathrm{Res}(f, \pm\sqrt{\mathfrak{a}^2-1}+i) = -\frac{i}{8\sqrt{\mathfrak{a}^2-1}}\sum_{\pm}\coth\left(\frac{\pm i+\sqrt{\mathfrak{a}^2-1}}{2\mathfrak{b}}\right), \tag{64}$$

$$\mathrm{Res}(f, 2n\pi\mathfrak{b}i) = \frac{4n\pi\mathfrak{b}^2 i}{(4n^2\pi^2\mathfrak{b}^2+\mathfrak{a}^2)^2 - 16n^2\pi^2\mathfrak{b}^2}, \tag{65}$$

and thus the contour integral reads

$$I_\gamma = \frac{\pi}{2\sqrt{\mathfrak{a}^2-1}}\frac{\sinh\frac{\sqrt{\mathfrak{a}^2-1}}{\mathfrak{b}}}{\cosh\frac{\sqrt{\mathfrak{a}^2-1}}{\mathfrak{b}} - \cos\frac{1}{\mathfrak{b}}} - \sum_{n=1}^{\Lambda}\frac{8n\pi^2\mathfrak{b}^2}{(4n^2\pi^2\mathfrak{b}^2+\mathfrak{a}^2)^2 - 16n^2\pi^2\mathfrak{b}^2}. \tag{66}$$

We will now consider the classical regime, which can be seen from fig. 6 to be where

$$2\pi\mathfrak{b} \gg \mathfrak{a}, \qquad \text{or equivalently} \qquad T \gg \frac{\hbar\Omega}{2\pi}, \tag{67}$$

which corresponds to the situation in which the classical poles are reached at much lower frequencies (in the complex plane) than the Matsubara modes. The classicality parameter (12) is given is by $\mathfrak{q} = 2\pi\mathfrak{b}/\mathfrak{a}$, so we can see the how the classicality is manifested in the approximation. It will be instructive to expand the above contour integral for $2\pi\mathfrak{b} \gg \mathfrak{a} > 1$,

$$I_\gamma \simeq \frac{\pi\mathfrak{b}}{\mathfrak{a}^2} + \frac{\pi}{12\mathfrak{b}} - \frac{\zeta(3)}{2\pi^2\mathfrak{b}^2} + \mathcal{O}\left(\frac{1}{\mathfrak{b}^3}\right), \tag{68}$$

which we can write as

$$\langle \delta T^2 \rangle \simeq \frac{T^2}{C_v} \left[1 + \frac{\hbar^2 \Omega^2}{12 T^2} - \frac{\hbar^3 \zeta(3) \Omega^2 \Gamma}{2\pi^3 T^3} + \mathcal{O}\left(\frac{\hbar^4 \Omega^2 \Gamma^2}{T^4} \right) \right]. \tag{69}$$

If $\mathfrak{b} \ll \sqrt{\mathfrak{a}^2 - 1}$, then the contour integral (66) can be approximated to be

$$I_\gamma \simeq \frac{\pi}{2\sqrt{\mathfrak{a}^2 - 1}} \left[\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{\mathfrak{a}^2 - 2}{2\sqrt{\mathfrak{a}^2 - 1}} \right) + \mathcal{O}\left(\frac{\mathfrak{b}^2}{\mathfrak{a}^4} \right) \right]. \tag{70}$$

If furthermore $\mathfrak{a} \gg 1$, the above expression simplifies as follows

$$I_\gamma \simeq \frac{\pi}{2\mathfrak{a}} - \frac{1}{\mathfrak{a}^2} + \mathcal{O}\left(\frac{1}{\mathfrak{a}^3}, \frac{\mathfrak{b}^2}{\mathfrak{a}^4} \right), \tag{71}$$

which we can write as

$$\langle \delta T^2 \rangle \simeq \frac{\hbar T \Omega}{2 C_v} - \frac{\hbar T \Gamma}{\pi C_v} + T \Omega \mathcal{O}\left(\frac{\Gamma^2}{\Omega^2}, \frac{T^2 \Gamma}{\Omega^3} \right). \tag{72}$$

When $\mathfrak{a} < 1$, the modes become those of the overdamped regime.

B Calculation of the scalar quasinormal modes of an AdS-Reissner-Nordström black hole

B.1 Boundary conditions

The equations of motion, which describe the coupled fluctuations of the metric and electromagnetic field on the background of a Reissner-Nordström black hole in 4-dimensional de Sitter space were obtained in [22] following the procedure outlined in [30]. The equations of motion for the AdS case can be obtained simply by considering a negative cosmological constant $\Lambda = -\frac{3}{R^2}$. The notation used in this section differs from the one used in the rest of the paper, but we adopt it in order to facilitate reference to [22, 30] keeping in mind that the final result (quasinormal frequencies) will be easy to interpret.

The background metric is

$$ds^2 = e^{2\nu} dt^2 - e^{2\psi} d\phi^2 - e^{2\mu_2} dr^2 - e^{2\mu_3} d\theta^2, \tag{73}$$

where

$$e^{2\nu} = \frac{\Delta}{r^2}, \quad e^{2\mu_2} = \frac{r^2}{\Delta}, \quad e^{2\mu_3} = r^2, \quad e^{2\psi} = r^2 \sin^2\theta, \tag{74}$$

and

$$\Delta = r^2 - 2Mr + Q^2 + \frac{r^4}{R^2}. \tag{75}$$

The background electromagnetic field is described by a single component of the field-strength tensor

$$F_{tr} = -\frac{Q}{r^2}. \tag{76}$$

The polar (even with respect to the change of sign of ϕ) mode involves the fluctuations of the metric $\delta\nu, \delta\mu_2, \delta\mu_3, \delta\psi$ and of the field-strength tensor $\delta F_{tr}, F_{t\theta}, F_{r\theta}$. The fluctuations with angular momentum l and the frequency ω may be described in the following way

$$\delta\nu = N(r)P_l(\theta), \qquad \delta F_{tr} = \frac{r^2 e^{2\nu}}{2Q} B_{tr}(r)P_l, \tag{77}$$

$$\delta\mu_2 = L(r)P_l(\theta), \qquad F_{t\theta} = -\frac{re^{\nu}}{2Q} B_{t\theta}P_{l,\theta}, \tag{78}$$

$$\delta\mu_3 = T(r)P_l + V(r)P_{l,\theta,\theta}, \qquad F_{r\theta} = -\frac{i\omega r e^{-\nu}}{2Q} B_{r\theta}(r)P_{l,\theta}, \tag{79}$$

$$\delta\psi = T(r)P_l + V(r)P_{l,\theta}\cot\theta.$$

Introducing the parameters

$$\mu^2 = 2n = (l-1)(l+2), \tag{80}$$

$$p_1 = 3M + (9M^2 + 4Q^2\mu^2)^{1/2}, \tag{81}$$

$$p_2 = 3M - (9M^2 + 4Q^2\mu^2)^{1/2},$$

one can rewrite the Einstein equations for fluctuations

$$\delta R_{ab} = -2\left[\eta^{nm}(\delta F_{an}F_{bm} + F_{an}\delta F_{bm}) - \eta_{ab}Q\delta F_{tr}/r^2\right] \tag{82}$$

as a system of differential equations of first order [22]

$$N_{,r} = aN + bL + c(nV - B_{r\theta}), \tag{83}$$

$$L_{,r} = \left(a - \frac{1}{r} + \nu_{,r}\right)N + \left(b - \frac{1}{r} - \nu_{,r}\right)L + c(nV - B_{r\theta}) - \frac{2}{r}B_{r\theta}, \tag{84}$$

$$nV_{,r} = -\left(a - \frac{1}{r} + \nu_{,r}\right)N - \left(b + \frac{1}{r} - 2\nu_{,r}\right)L - \left(c + \frac{1}{r} - \nu_{,r}\right)(nV - B_{r\theta}) + B_{t\theta}, \tag{85}$$

$$B_{t\theta} = B_{r\theta,r} + \frac{2}{r}B_{r\theta}, \tag{86}$$

$$r^4 e^{2\nu}B_{tr} = 2Q^2\left[2T - l(l+1)V\right] - l(l+1)r^2 B_{r\theta}, \tag{87}$$

$$(r^2e^{2\nu}B_{t\theta})_{,r} + r^2e^{2\nu}B_{tr} + \omega^2r^2e^{-2\nu}B_{r\theta} = 2Q^2\frac{N+L}{r}, \tag{88}$$

where

$$a = \frac{n+1}{r}e^{-2\nu}, \tag{89}$$

$$b = -\frac{1}{r} + \nu_{,r} + r\nu_{,r}^2 + \omega^2e^{-4\nu}r - 2\frac{e^{-2\nu}}{r^3}Q^2 - \frac{ne^{-2\nu}}{r}, \tag{90}$$

$$c = -\frac{1}{r} + r\nu_{,r}^2 + \omega^2e^{-4\nu}r - \frac{2e^{-2\nu}}{r^3}Q^2 + \frac{e^{-2\nu}}{r}. \tag{91}$$

These equations can be decoupled upon introducing the functions

$$Z_1^+ = p_1H_1^+ + (-p_1p_2)^{1/2}H_2^+, \tag{92}$$

$$Z_2^+ = -(-p_1p_2)^{1/2}H_1^+ + p_1H_2^+, \tag{93}$$

where

$$H_1^+ = -\frac{1}{Q\mu}\left[r^2 B_{r\theta} + 2Q^2\frac{r}{\bar{\omega}}(L + nV - B_{r\theta})\right], \tag{94}$$

$$H_2^+ = rV - \frac{r^2}{\bar{\omega}}(L + nV - B_{r\theta}), \tag{95}$$

and

$$\bar{\omega} = nr + 3M - \frac{2Q^2}{r}. \tag{96}$$

The system can be reduced to the couple of equations

$$\frac{\Delta}{r^2}\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{d}{dr}Z_i^+\right) + \omega^2 Z_i^+ = V_i^+ Z_i^+ \qquad (i = 1, 2) \tag{97}$$

with Schrödinger-type potentials

$$V_1^+ = \frac{\Delta}{r^5}\left[U + \frac{1}{2}(p_1 - p_2)W\right], \tag{98}$$

$$V_2^+ = \frac{\Delta}{r^5}\left[U - \frac{1}{2}(p_1 - p_2)W\right], \tag{99}$$

where

$$U = (2nr + 3M)\,W + \left(\bar{\omega} - nr - M + 2\frac{r^3}{R^2}\right) - \frac{2nr^2}{\bar{\omega}}e^{2\nu}, \tag{100}$$

$$W = \frac{\Delta}{r\bar{\omega}^2}(2nr + 3M) + \frac{1}{\bar{\omega}}\left(nr + M - 2\frac{r^3}{R^2}\right). \tag{101}$$

At this point it is useful to note, that for vanishing charge ($Q \to 0$) the potential V_2^+ reduces to the potential for purely gravitational polar fluctuations of the Schwarzschild black hole in AdS [20]. It is not surprising, because in this limit p_2 vanishes and hence Z_2^+ reduces to purely gravitational fluctuations H_2^+ plus a term proportional to $B_{r\theta}$, which vanishes itself according to the definition $B_{r\theta} \sim QF_{r\theta}$ (79). Similarly, in this limit the mode described by Z_1^+ corresponds to purely electromagnetic fluctuations on the background of the neutral black hole. Keeping these connections in mind at nonzero Q, we will still call the modes associated with Z_1^+ and Z_2^+ "electromagnetic" and "gravitational", respectively.

In what follows, we will consider the fluctuations described by Z_i^+ and recover the asymptotic behavior of the metric fields in this mode. In order to do this, we need to complete the solution of (83-88) following the procedure outlined in [30]. First of all, we note that the sum of (84) and (85) after the substitution of (86) may be written as

$$L_{,r} + \left(\frac{2}{r} - \nu_{,r}\right)L = -\left[X_{,r} + \left(\frac{1}{r} - \nu_{,r}\right)X\right], \tag{102}$$

where

$$X = nV - B_{r\theta}. \tag{103}$$

On the other hand, we notice that the linear combination of H_1^+ and H_2^+, which we denote by Z^*, is expressed in terms of L and X as

$$\begin{aligned} Z^* = nH_2^+ + \frac{Q\mu}{r}H_1^+ &= \frac{1}{\bar{\omega}}(3Mr - 4Q^2)X - \frac{1}{\bar{\omega}}(nr^2 + 2Q^2)L \\ &= rX - \frac{nr^2 + 2Q^2}{\bar{\omega}}(L + X). \end{aligned} \tag{104}$$

Substituting X from this expression into (102) we get

$$\bar{\omega}\frac{d}{dr}\left(\frac{r^3e^{-\nu}}{3Mr-4Q^2}L\right) = -r\frac{d}{dr}\left(re^{-\nu}\frac{\bar{\omega}}{3Mr-4Q^2}Z^*\right). \tag{105}$$

The expression for L is thus obtained via the integral

$$\begin{aligned} L &= -\frac{3Mr-4Q^2}{r^3e^{-\nu}}\int dr\frac{r}{\bar{\omega}}\frac{d}{dr}\left(re^{-\nu}\frac{\bar{\omega}}{3Mr-4Q^2}Z^*\right) \\ &= -\frac{1}{r}Z^* + \frac{3Mr-4Q^2}{r^3e^{-\nu}}\left[\int dr\frac{e^{-\nu}}{\bar{\omega}}Z^* + C\right], \end{aligned} \tag{106}$$

where in the second line we have performed an integration by parts and C is an integration constant. However, the expression for X can similarly be obtained by substituting L of (104) into (102). After the integration by parts it reads

$$X = \frac{1}{r}Z^* + \frac{nr^2+2Q^2}{r^3e^{-\nu}}\left[\int dr\frac{e^{-\nu}}{\bar{\omega}}Z^* + C\right]. \tag{107}$$

One can check, that the constants of integration in (106) and (107) are consistent by plugging these expressions into (102). We notice that the sum of L and X assumes a concise form

$$L + X = \frac{\bar{\omega}}{r^2e^{-\nu}}\left[\int dr\frac{e^{-\nu}}{\bar{\omega}}Z^* + C\right]. \tag{108}$$

To proceed with the evaluation of N, we take the derivative (104) and substitute the expression of $(L+X)_{,r}$ from (102)

$$Z^*_{,r} = rX_{,r} + \left(\frac{3Mr-4Q^2}{r\bar{\omega}}\right)X - r^2e^{-\nu}\frac{d}{dr}\left(\frac{1}{r^2e^{-\nu}}\frac{nr^2+2Q^2}{\bar{\omega}}\right)(L+X). \tag{109}$$

Finally, we use equations (85) and (86) to eliminate $X_{,r}$ and obtain the expression for N

$$\begin{aligned} N = \frac{1}{ra-1+r\nu_{,r}}\Bigg\{&\frac{2}{r}B_{r\theta} - \frac{d}{dr}Z^* - \\ &-\left[rb+1-2r\nu_{,r}+r^2e^{-\nu}\frac{d}{dr}\left(\frac{1}{r^2e^{-\nu}}\frac{nr^2+2Q^2}{\bar{\omega}}\right)\right]\frac{\bar{\omega}}{r^2e^{-\nu}}\left(\int dr\frac{e^{-\nu}}{\bar{\omega}}Z^* + C\right) + \\ &+\left(\frac{3Mr-4Q^2}{r\bar{\omega}} - ar\right)\left[\frac{1}{r}Z^* + \frac{nr^2+2Q^2}{r^3e^{-\nu}}\left(\int dr\frac{e^{-\nu}}{\bar{\omega}}Z^* + C\right)\right]\Bigg\}. \end{aligned} \tag{110}$$

Let us now turn to the asymptotic behavior of the master functions Z_1^+ and Z_2^+. The corresponding Schrödinger equations at large r take the form

$$\left(\frac{r^2}{R^4}\partial_r r^2\partial_r + \omega^2 - \frac{\mu^2+2}{R^2} - \frac{2p_1^2}{\mu^4R^4}\right)Z_2^+ = 0, \tag{111}$$

$$\left(\frac{r^2}{R^4}\partial_r r^2\partial_r + \omega^2 - \frac{\mu^2+2}{R^2} - \frac{2p_2^2}{\mu^4R^4}\right)Z_1^+ = 0. \tag{112}$$

As expected in the limit $Q \to 0$ the equation for Z_2^+ reduces to that of the Schwarzschild black hole in [14]. Hence, for large r, the asymptotics of the master functions can be expressed as sums of linearly independent modes

$$Z_2^+\Big|_{r\to\infty} = \alpha + \frac{\beta}{r},$$
$$Z_1^+\Big|_{r\to\infty} = \gamma + \frac{\delta}{r}.$$

From (92), (93) we get the expressions for H_1^+ and H_2^+ and from (104) the asymptotic expression for Z^*

$$Z^* = \xi + \frac{\eta}{r} + \frac{\zeta}{r^2}, \tag{113}$$

where

$$\xi = \frac{n\big(p_1\alpha + \sqrt{-p_1p_2}\;\gamma\big)}{p_1(p_1-p_2)}, \tag{114}$$
$$\eta = \frac{n\big(p_1\beta + \sqrt{-p_1p_2}\;\delta\big) + Q\mu\big(p_1\gamma - \sqrt{-p_1p_2}\;\alpha\big)}{p_1(p_1-p_2)}, \tag{115}$$
$$\zeta = \frac{Q\mu\big(p_1\delta - \sqrt{-p_1p_2}\;\beta\big)}{p_1(p_1-p_2)}. \tag{116}$$

After plugging this expansion into (106), (107) and (110) we get at large r

$$L\Big|_{r\to\infty} = -\frac{\xi - \frac{3M}{R}C}{r} - \frac{\eta + \frac{3M}{n}\xi + 4C\frac{Q^2}{R}}{r^2} + O\left(\frac{1}{r^3}\right), \tag{117}$$
$$V\Big|_{r\to\infty} = \frac{C}{R} + \frac{\eta + \frac{3M}{n}\xi + C\left(4\frac{Q^2}{R} + n\right)}{2nr^2} + O\left(\frac{1}{r^3}\right), \tag{118}$$
$$N\Big|_{r\to\infty} = -\frac{\eta + \frac{3M}{n}\xi - CnR^3\omega^2}{r^2} + O\left(\frac{1}{r^3}\right). \tag{119}$$

We should note here that because $F_{r\theta}$ behaves as r^{-2} on the boundary (one can see this from the $r \to \infty$ expansion of the Maxwell equations) the function $B_{r\theta}$ which enters the definition of X (103) and the expression for N (110) falls off as r^{-3} and does not enter the above expansions. As discussed in [14], the perturbations of the metric near the AdS boundary have two linearly independent modes, which behave as r^2 and $\frac{1}{r}$. The former violates the asymptotic behavior of the AdS metric on the boundary and should be forbidden. The latter can be nicely interpreted in the AdS/CFT correspondence as a vacuum expectation value of the stress-energy tensor of the dual field theory and thus we need to keep it. According to the definitions (73), (77–79) keeping the mode $\sim r^{-1}$ in the boundary metric fluctuations, means keeping only the mode $\sim r^{-3}$ in the functions N and V. Hence, we need to choose the constant of integration C equal to zero and demand the Robin boundary conditions on the wave function Z^*

$$\eta = -\frac{3M}{n}\xi. \tag{120}$$

Taking into account the definitions (114) we can derive the conditions on the master wave functions Z_1^+ and Z_2^+. Because the equations (97) are independent we can consider the "gravitational" and "electromagnetic" modes separately. Thus we find

$$\beta = -\left(\frac{3M}{n} + \frac{4Q^2}{p_1}\right)\alpha, \qquad \text{when } Z_1^+ = 0; \tag{121}$$

$$\delta = -\left(\frac{3M}{n} - \frac{p_1}{2n}\right)\gamma, \qquad \text{when } Z_2^+ = 0. \tag{122}$$

We note that to obtain this result one needs to take the negative branch of the square root in (81): $\sqrt{4\mu^2 Q^2} = -2\mu Q$. This choice is motivated by the comparison with the hydrodynamic treatment (33), discussed previously. Taking the positive branch would give results which are inconsistent with hydrodynamics, so we ignore this possibility as unphysical. One can check, that in the limit $Q \to 0$ the first of these conditions coincides with that obtained for the Schwarzschild black hole in [14]. This is consistent with the fact pointed out earlier, that the Z_2^+ mode becomes purely gravitational in this limit. The second condition vanishes in this case, because at $Q = 0$ the "electromagnetic" mode does not couple to gravity and the treatment based on the asymptotic behavior of the metric is no longer valid.

The boundary conditions at the horizon $r = r_+$ are easier to obtain. By definition, the quasinormal mode should contain only the wave "infalling" to the horizon. In "tortoise" coordinates $dr^* = \frac{r^2}{\Delta} dr$ the Schrödinger equation (97) takes the simple form

$$\left[\partial_{r_*}^2 - \partial_\tau^2 - V_i^+\right] Z_i^+ = 0, (i = 1, 2) \tag{123}$$

Noticing that V_i^+ vanishes at the horizon, we get the infalling-wave solution in the form

$$Z_i^+\Big|_{r_* \to -\infty} \sim e^{-i\omega(\tau + r_*)}, \tag{124}$$

B.2 Numerical solution

In order to proceed with the numerical calculation of the quasinormal modes, we make several redefinitions of variables. First of all we substitute the infalling-wave Ansatz

$$Z = e^{-i\omega(\tau + r_*)}\psi(r), \tag{125}$$

and get the equation for ψ

$$\psi''(r) + \left[\frac{r^2}{\Delta}\frac{d}{dr}\frac{\Delta}{r^2} - 2i\omega\frac{r^2}{\Delta}\right]\psi'(r) - \frac{r^4}{\Delta^2}V_i^+\psi(r) = 0. \tag{126}$$

To compactify the interval of integration we introduce the variable $y = 1 - \frac{r_+}{r}$. After this substitution the boundary of AdS is located at $y = 1$ and the horizon is at $y = 0$. The boundary conditions for $\psi(y)$ can be easily derived from (121) and (124). At the horizon the infalling-wave boundary condition is simply stated as

$$\psi(0) = 1. \tag{127}$$

On the AdS boundary, the condition is found from the expansion of (125) at $r \to \infty$

$$\psi(y)\Big|_{y \to 1} = 1 + \frac{1}{r_+}\left(\frac{3M}{n} + \frac{4Q^2}{p_1} + i\omega\right)(y - 1) + \ldots \qquad \text{for "gravitational" mode,} \tag{128}$$

$$\psi(y)\Big|_{y \to 1} = 1 + \frac{1}{r_+}\left(\frac{3M}{n} - \frac{p_1}{2n} + i\omega\right)(y - 1) + \ldots \qquad \text{for "electromagnetic" mode.} \tag{129}$$

Similar to [14] we use these boundary conditions to expand the solution in series around the singular points of (126) at $y = 0$ and $y = 1$ to sufficiently high order and then solve the equation numerically by seeding the shooting procedure from both ends of the interval. We then look for a frequency ω_0 at which the Wronskian of the two solutions coming from opposite ends is zero at an intermediate point. This tells us that at that given frequency the shooting solutions can be smoothly connected, resulting in a nontrivial solution to (126) on the full interval with boundary conditions (127),(128). This solution is the quasinormal mode and the frequency ω_0 is the quasinormal frequency of the black hole.

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